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Conduction through a grooved surface and Sierpinsky fractals

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Abstract

Conduction in a semi-infinite wall with a grooved line of contact between the wall material and convective environment is studied using series expansions. A periodic composition of semicircles is shown to result in a uniform gradient distribution at specific values of the groove radius and the convection heat transfer coefficient. Two fractal parquets exposed to natural thermal gradients are studied by the methods of complex analysis. In double periodic patterns each elementary cell is fractal (Sierpinsky's carpet and Sierpinsky's gasket) in which 'dark' and 'light' phases have arbitrary conductivities. The Maxwell approximation is used to calculate effective characteristics of both fractal structures by 'homogenization' of the environment of an 'inclusion'. Solution of an exact two-dimensional refraction problem within an elementary cell including two components is used for upscaling, i.e. recalculation of effective conductivities and dissipations of subfractals of consequently increasing order. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Problems of heat conduction in domains with interfaces between different phases include two juxtaposing areas: heterogeneous media (spatially-varying conductivity) and non-smooth outer boundaries of domains of homogeneous materials, in particular, developed surfaces. Such extended surfaces allow to enhance the cooling of walls. Since extensive reviews are available on both subjects [1,2], the survey of the literature below is reduced to a minimum.

In what follows, we develop further our analytical approach to study steady, two-dimensional (2D) heat conduction [3,4]. The goals of this note are twofold. First, we show that for a specific value of the heat convection transfer coefficient, the temperature gradient is constant within a wall which surface is extended by a periodic system of semi-circular troughs. Second, we apply the analytical solutions of the R-linear conjugation problem [5] to calculate the effective conductivity and dissipation of two double periodic fractal structures.

2. Uniform heat flux from a periodic system of troughs

Consider a semi-infinite wall of conductivity k whose surface is extended by a system of semi-circular troughs of radius a (Fig. 1).

Steady state temperature $T(x, y)$ within the wall

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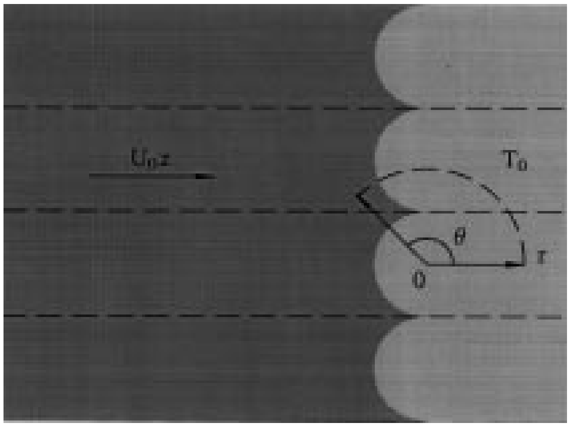

Fig. 1. System of semi-circular troughs of radius a.

satisfies the Laplace equation. Temperature gradient at infinity, U_0, is perpendicular to the mean line of the wall surface around which the grooves wind.

By virtue of symmetry split off one element of the system, a half-strip with a rounded side. For definiteness, consider an element in Fig. 1, where the dashed lines represent the two boundaries of the element through which it is periodically continued upwards and downwards. At the wall surface the convection surface condition [6, p.63] holds:

$$k\frac{\partial T}{\partial n} = -\alpha(T - T_0) \tag{1}$$

where k is the wall's conductivity, α is the convection heat transfer coefficient and $\partial T/\partial n$ is the normal derivative of temperature in the direction of the outward normal to the surface.

We show that for the system described above the wall temperature gradient will be uniform under some additional conditions. To prove this statement consider a circular hole in an infinite medium of constant conductivity k. The pattern differs from the classical cylindrical inhomogeneity of conductivity k_c [7 pp. 426–428] in the boundary conditions along the circumference. Namely, instead of matching temperature and the flux normal components (so called fourth-type boundary condition) along the interface, we impose the third type boundary condition (1).

In the cylindrical coordinate system originating from the centre of the inclusion (O in Fig. 1) a suitable form for the steady temperature distribution outside the cylinder is [8, Ch. VI]:

$$T(r, \theta) = T_0 + \sum_{n=1}^{\infty} A_n r^{-n} \cos n\theta + U_0 r \cos\theta \tag{2}$$

where r and θ are the radial and angular coordinates.

The coefficients of expansion A_n are determined from Eq. (1) routinely and after some algebra we get:

$$A_1 = U_0 a^2 \frac{k - \alpha a}{\alpha a + k}, \qquad A_2 = A_3 = \cdots = 0 \tag{3}$$

Notice that we can relate this solution with one for the conjugation problem mentioned [7]. In particular, from Eq. (3) it follows that $\alpha = k_c/(ka)$, where k_c is conductivity value of the cylinder in the problem with fourth-type boundary conditions. Similar comparisons are common in ground-water hydrology where the third-type boundary condition coefficients (the Mjatiev–Girinskii model) in Eq. (1) are expressed through the values of thickness and conductivity of the thin transition zone (so called aquitard). In this zone flow is predominately orthogonal to the interface [3].

Let us choose now $a = k/\alpha$ in Eq. (3). Then all coefficients in Eq. (2) are zero and, hence, the temperature gradient equals U_0 within the wall. Due to uniformity of U we can now attach the circles of radius a to each other periodically. Then, due to symmetry we remove either right- or left-hand side of the resulting picture. Consequently, we come to the developed surface in Fig. 1.

Even though the case $a = k/\alpha$ is very specific, it

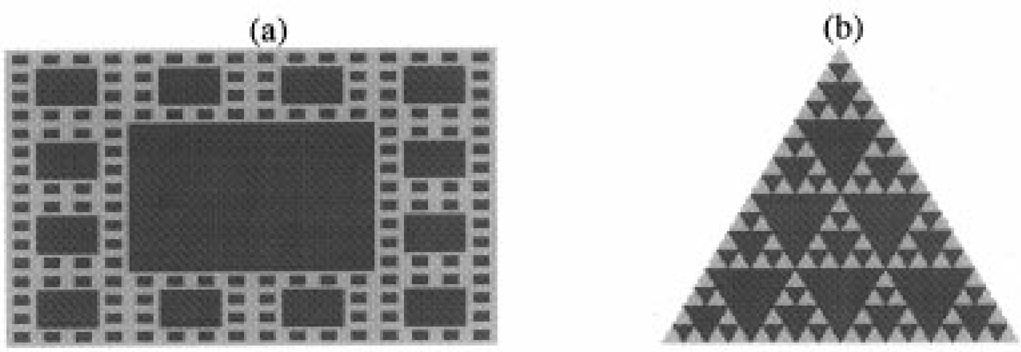

Fig. 2. Sierpinsky's carpet (a), and Sierpinsky's gasket (b); subfractals of third order.

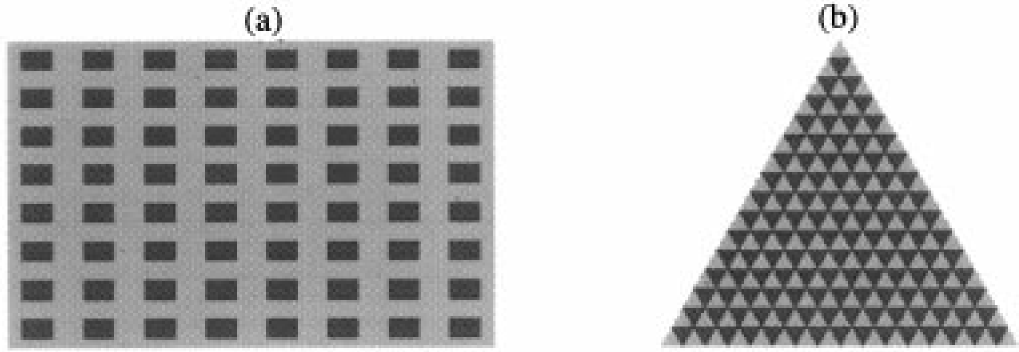

Fig. 3. An array of rectangular inclusions (ARI) (a), and a regular triangular checkerboard structure (TCS) (b).

shows that strictly uniform gradient distributions can appear in walls with developed surfaces, the fact rarely emphasized in textbooks.

3. Heat conduction in two fractal structures

Consider two-phase media (parquets) composed of 'dark' and 'light' components with arbitrary conductivities k_1 and k_2, respectively. The structures are placed in an outer temperature field. Regular (non-fractal) parquets were studied by Obnosov [5,9–11] and several non-trivial features (not distinguishable by standard numerical or homogenization techniques) were described in an explicit, analytical form. Here, we study fractal parquets, where the known Sierpinsky patterns repeat themselves periodically in two directions. Such structures are now studied in many branches of continuum mechanics [12], in particular, in heat conduction [13]. However, to our knowledge, there are no explicit solutions based on 2D temperature distributions taking into account the temperature field refraction along the two component interfaces for arbitrary k_1 and k_2.

Fig. 2 shows subfractals generated from two double-periodic structures: (a) an array of rectangular inclusions (ARI) shown in Fig. 3a; (b) a regular triangular checkerboard structure (TCS) shown in Fig. 3b. The procedure of fractal generation is straightforward. For example, in Fig. 3a we split off an elementary rectangular cell $2l \times 2h$ (Fig. 4a), the 'dark' rectangle inclusion being of length l and height h. For this cell an exact temperature field was analysed [5,9]. Next step generates 12 smaller rectangles ($l/4 \times h/4$) within the elementary cell, then 144 rectangles ($l/16 \times h/16$) are generated, etc. The procedure is carried on within all elementary cells in Fig. 3a and a double-periodic composition of these structures constitutes the final fractal parquet. The second and third order subfractals are shown in Figs. 5a and 2a correspondingly. They represent the plane covered by a patching of Sierpinsky's carpets.

Figs. 3b, 3b and 5b demonstrate the stages of construction of fractal structures (parquets composed by

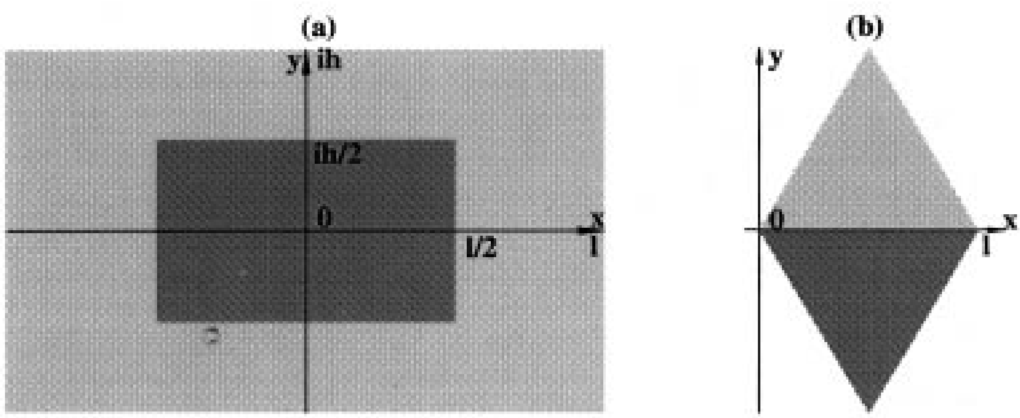

Fig. 4. An elementary cell of ARI (a), and TCS (b).

Sierpinsky's gaskets) on the base of the elementary cell shown in Fig. 4b. This parquet was studied in Ref. [12] by a type of homogenization method.

Return to the subfractals of first order (Fig. 3). Consider the heat flux $\mathbf{U}(x, y) = (U_x(x, y), U_y(x, y))$, where $U_x(x, y)$ and $U_y(x, y)$ are the horizontal and vertical flux components, respectively. In each of the two phases D_1, D_2 the Fourier law holds:

$$\mathbf{U}_j(x, y) = -k_j \nabla T_j(x, y), \quad (x, y) \in D_j, \ j = 1, 2$$

Temperature $T(x, y)$ is harmonic within the two components of the parquet. The flux $\mathbf{U}(x, y)$ is interpreted as a complex function $\mathbf{U}(z) = U_x(x, y) + iU_y(x, y)$ of the complex coordinate $z = x + iy$. $\mathbf{U}(z)$ is an antiholomorphic function [14, pp. 73–75] within the 'dark' and 'light' phases i.e. $\mathbf{U}(z) = \overline{\mathrm{d}T(z)/\mathrm{d}z}$, where overbarring means complex conjugation. Two standard conditions hold along the interfaces L:

$$\left[\mathbf{U}_1(t)\right]_n = \left[\mathbf{U}_2(t)\right]_n,$$

$$\left[\frac{1}{k_1}\mathbf{U}_2(t)\right]_\tau = \left[\frac{1}{k_2}\mathbf{U}_2(t)\right]_\tau, \quad t \in L$$

i.e continuity of the normal component and jump of the tangential component of $\mathbf{U}(z)$.

Exact solutions for both periodic structures in Fig. 3 were developed in Refs. [10,11]. In particular, in Ref. [5] the effective thermal conductivity

$$k_{\mathrm{ef}} = \frac{\langle k(z)\mathbf{U}(z)\rangle}{\langle \mathbf{U}(z)\rangle} \tag{4}$$

and dissipation

$$D = \langle |\mathbf{U}(z)|^2 / k(z)\rangle \tag{5}$$

were found. The angular brackets here designate averaging over the elementary cell S i.e.

$$\langle \mathbf{U}(z)\rangle = \frac{1}{|S|}\int_S \mathbf{U}(z)\,\mathrm{d}s$$

$|S|$ is the area of the cell S.

For ARI the functionals (4), (5) are

$$k_{\mathrm{ef}} = k_{\mathrm{ef}}(k_1, k_2) = k_1\sqrt{\frac{2+\Delta}{2-\Delta}}\frac{Q_x + iQ_y}{Q_x\vartheta_1/\vartheta + iQ_y\vartheta/\vartheta_1} \tag{6}$$

$$D = D(k_1, k_2) = \frac{1}{k_1}\sqrt{\frac{2-\Delta}{2+\Delta}}\left(Q_x^2\frac{\vartheta_1}{\vartheta} + Q_y^2\frac{\vartheta}{\vartheta_1}\right) \tag{7}$$

where

$$\vartheta = \vartheta(m) = \frac{F(1/2, 1/2; 1; m)}{F\left(\frac{1+\lambda}{2}, \frac{1-\lambda}{2}; 1; m\right)}, \qquad \vartheta_1 = \vartheta(1-m) \tag{8}$$

$$\lambda = \frac{2}{\pi}\arcsin\frac{|\Delta|}{2}, \qquad \Delta = \frac{k_2 - k_1}{k_2 + k_1} \tag{9}$$

In its turn, $F(\alpha; \beta; \gamma; z) = {}_2F_1(\alpha, \beta; \gamma; z)$ is hypergeometric function. The parameter $m = m(h/l)$ is calculated from the 'dark' rectangle's side ratio (l/h) [15, Table 17.3]. $Q_0 = Q_x + iQ_y$ and Q_x, Q_y are the fluxes through adjoint sides of the corresponding elementary cell:

$$Q_x = \frac{1}{2h}\int_{-h}^{h} \mathrm{Re}\mathbf{U}(l + iy)\,\mathrm{d}y,$$

$$Q_y = \frac{1}{2l}\int_{-l}^{l} \mathrm{Im}\mathbf{U}(x + ih)\,\mathrm{d}x$$

For TCS

$$k_{\mathrm{ef}} = \sqrt{k_1 k_2}, \qquad D = \frac{1}{\sqrt{k_1 k_2}}|Q_0|^2 \tag{10}$$

where $Q_0 = (2Q_2 + Q_1)/\sqrt{3} + iQ_1$ and

$$Q_1 = \frac{1}{l}\int_0^l \mathrm{Im}\mathbf{U}(x)\,\mathrm{d}x, \qquad Q_2 = \frac{1}{l}\int_0^l \mathrm{Re}\left[\mathrm{e}^{\mathrm{i}\pi/6}\mathbf{U}(\mathrm{e}^{\mathrm{i}\pi/3}x)\right]\mathrm{d}x$$

Now we apply the rigorous results valid for the subfractals of first order (Fig. 3) to the case of arbitrary order fractals. For this purpose, treating both structures in Fig. 2, we calculate approximate values of k_{ef} and D for subfractals of nth order. For step-by-step approximations we follow the Maxwell procedure, i.e at each step we substitute the 'environment' of the corresponding 'dark' element by an effectively homogeneous medium. Recall that Maxwell substituted the environment (generally inhomogeneous) of a spherical inclusion by an effectively homogeneous medium and used an exact solution with conjugation conditions along the sphere surface. His approximation is valid for sufficiently 'diluted' suspensions, i.e. the distance between two neighbouring inclusions should be sufficiently large. Note, that the Maxwell approximation resulted in relatively simple formulae because the flux within each sphere (and generally for any ellipsoidal inclusion) is constant.

Thus, in our case for any subfractal the outer medium within any elementary cell is assumed to be effectively homogeneous (clearly, in reality it is not homogeneous and the distance between two neighbouring 'dark' inclusions is not too high). Then, the effective parameters are calculated on the base of the rigorous

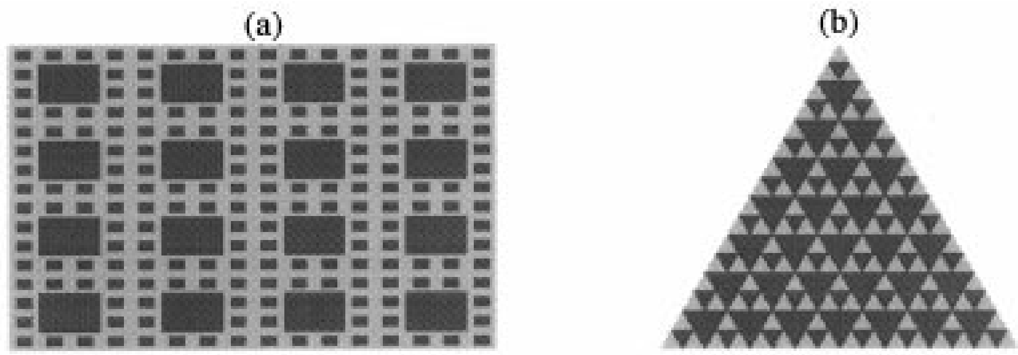

Fig. 5. Sierpinsky's carpet (a), and Sierpinsky's gasket (b); subfractals of second order.

solutions for the subfractal of first order (presented above). Emphasize, that the temperature distribution both in the 'environment' and in the inclusion are essentially 2D (the flux is not constant). Then upscaling is performed and for the 'dark' rectangle of larger dimension the procedure is repeated until the whole medium scale (nth step) is reached.

For simplicity, assume that the external field is directed along one of the symmetry axes ($Q_0 = Q_x$, $Q_y = 0$ or $Q_x = 0$, $Q_0 = iQ_y$) in the case of ARI. For TCS, Q_0 is oriented arbitrary. Then, 'beginning from the end' we can evaluate dissipation $D^1 = D(k_1, k_2)$ of the structures shown in Fig. 3 using Eqs. (6)–(10).

Designate $k_1 = k_1^1$ and $k_1^2 = k_{ef}(k_1^1, k_2)$. Then we calculate the dissipation of the subfractals (Fig. 5) of the second order as $D^2 = D(k_1^2, k_2)$. Continuing this evaluation, at the nth step, we get

$$D^n = D\left(k_1^n, k_2\right), \quad \text{where } k_1^n = k_{ef}\left(k_1^{n-1}, k_2\right)$$

Fig. 6 shows the results of calculation of the effective conductivity of the structure in Fig. 2a as a function of k_2 for subfractals up to the ninth order at $k_1 = k_1^1 = 1$ and $l = h$. Clearly, the graphs show that k_1^n tends to k_2 when $n \to \infty$.

For the case of an array of square inclusions (ARI with $l = h$) and an arbitrary TCS, dissipation is $1/k_1^n$ multiplied by $|Q_0|^2$. Besides, D does not depend on the orientation of the external field (i.e. the argument of the vector Q_0).

4. Conclusion

We have used the standard interface condition (1) for a developed surface. As was discussed earlier [3]

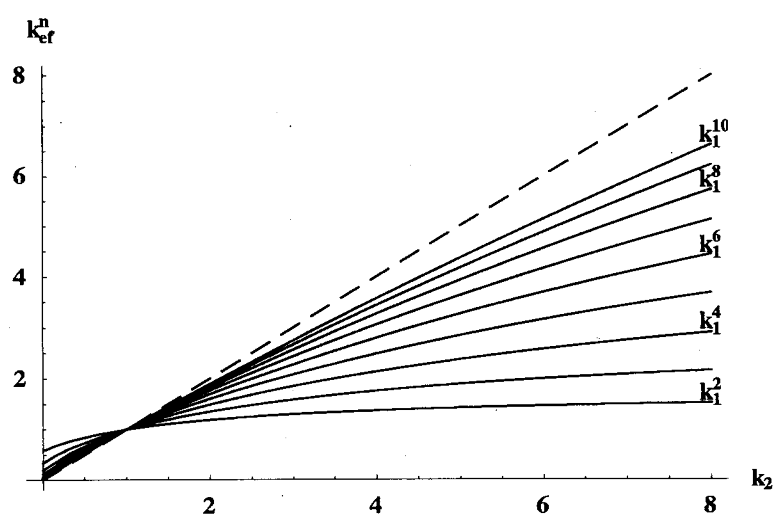

Fig. 6. Effective thermal conductivities of Sierpinsky's square carpet; subfractals from first to ninth order.

solutions of boundary value problems with this condition can be compared with solutions involving physically better substantiated refraction conditions. For the system of grooves studied the results of [16] can be readily applied.

The 'homogenization' used for determination of effective parameters of fractals is essentially different from usually performed. Forsooth, we do not obliterate refraction of the temperature field along interfaces and involve the exact solution of the conjugation problem at each step of fractalization. Moreover, from Eq. (6) the effective conductivity of the first order subfractal of ARI is generally a tensor except for a very specific case discussed above ($l = h, Q_0 = Q_x$ or $Q_0 = iQ_y$). Apparently, D is scalar for any parquet. Emphasize, that in many standard simplified homogenization techniques the effective conductivity is incorrectly postulated to be a scalar. Moreover, in the literature we could not find veracious proofs that upscaling always results in an 'ellipse' of effective conductivity. Such an ellipse became a synecdoche of the macroconductivity of heterogeneous media though upon scrupulous averaging [5] even seemingly 'obvious' statements of Dykhne [17,18] occurred to be inadequate for some parquets.

For general ARI, evaluation of the effective conductivity of subfractals of orders higher than one calls for consequential upscaling of tensorial values. Further, we plan to determine these tensors. We intend to utilize the solution for ARI and other rigorous solutions in thermal contact problems. In particular, we are going to model the contact zone either as an array of 'bubbles' or as a rippled, zigzag interface (see [19]).

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